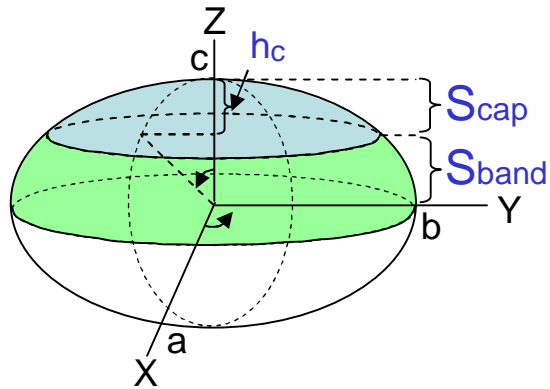


一般楕円体の部分表面積を数値積分で求める



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

| | | | | | |
|----------|-------------------|---|---------|-------|-----|
| 帯領域の表面積 | S_{band} | 0 | z | h_b | c |
| 帽子領域の表面積 | S_{cap} | 0 | $c-h_c$ | z | c |

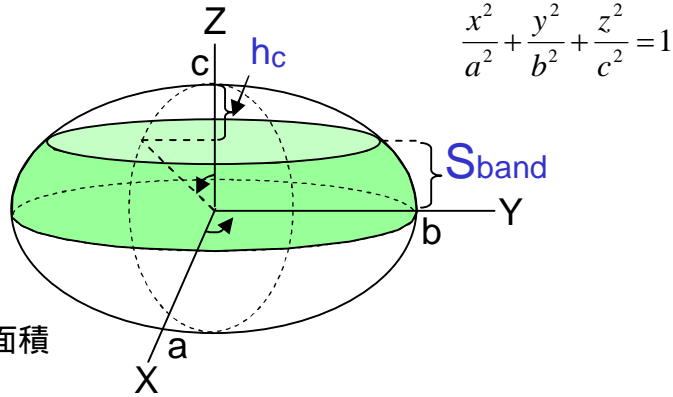
Gauss-Chebyshev integration

$$I = \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \sum_{i=1}^n w_i f(x_i)$$

$$w_i = \frac{\pi}{n}, \quad x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$$

- 1) 複素数計算により、a,b,cの大きさに制限なく求める
- 2) h_c, h_b が微小値でも $S_{\text{cap}}, S_{\text{band}}$ の精度を確保する (keisan上の演算のみ有効)
- 3) Gauss-Chebyshev積分を利用する (分点、重みの計算が容易)

楕円体の冠帯部の表面積を数値積分で求める



冠帯部の表面積

$$\begin{aligned}
 S_{band} &= \iint dS \\
 &= \int_{\theta=\cos^{-1}h}^{\frac{\pi}{2}} \int_{\varphi=0}^{2\pi} \sqrt{b^2c^2 \sin^2 \theta \cos^2 \varphi + c^2a^2 \sin^2 \theta \sin^2 \varphi + a^2b^2 \cos^2 \theta} \sin \theta d\theta d\varphi \\
 &= 4bc \int_{\varphi=0}^{\frac{\pi}{2}} \sqrt{B} \int_{u=0}^h \sqrt{1+Fu^2} du d\varphi \\
 &= 2bc \int_0^{\frac{\pi}{2}} \sqrt{B} \left[u\sqrt{1+Fu^2} + \frac{1}{\sqrt{F}} \sinh^{-1}(\sqrt{Fu}) \right]_0^h d\varphi \\
 &= 2bch \int_0^{\frac{\pi}{2}} \sqrt{B} \left(\sqrt{1+Fh^2} + \frac{\sinh^{-1}(\sqrt{Fh})}{\sqrt{Fh}} \right) d\varphi \\
 &= bch \int_{-1}^1 \sqrt{B(t)} \left(\sqrt{1+F(t)h^2} + \frac{\sinh^{-1}(h\sqrt{F(t)})}{h\sqrt{F(t)}} \right) \frac{dt}{\sqrt{1-t^2}} \\
 &= bch \sum_{i=1}^n \frac{\pi}{n} f \left(\cos \left(\frac{2i-1}{2n} \pi \right) \right)
 \end{aligned}$$

Gauss-Chebyshev integration

$$S = bch \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = bch \sum_{i=1}^n w_i f(x_i) \quad w_i = \frac{\pi}{n}, \quad x_i = \cos \left(\frac{2i-1}{2n} \pi \right)$$

$$x = a \sin \theta \cos \varphi$$

$$y = b \sin \theta \sin \varphi$$

$$z = c \cos \theta$$

$$\mathbf{r} = a \sin \theta \cos \varphi \mathbf{i} + b \sin \theta \sin \varphi \mathbf{j} + c \cos \theta \mathbf{k}$$

$$d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial \theta} d\theta \times \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \theta \cos \varphi & b \cos \theta \sin \varphi & -c \sin \theta \\ -a \sin \theta \sin \varphi & b \sin \theta \cos \varphi & 0 \end{vmatrix} d\theta d\varphi$$

$$= (bc \sin \theta \cos \varphi \mathbf{i} + ca \sin \theta \sin \varphi \mathbf{j} + ab \cos \theta \mathbf{k}) \sin \theta d\theta d\varphi$$

$$dS = |d\mathbf{S}| = \sqrt{b^2c^2 \sin^2 \theta \cos^2 \varphi + c^2a^2 \sin^2 \theta \sin^2 \varphi + a^2b^2 \cos^2 \theta} \sin \theta d\theta d\varphi$$

$$u = \cos \theta, \quad du = -\sin \theta d\theta, \quad h = \frac{h_b}{c}$$

$$B = \cos^2 \varphi + \frac{a^2}{b^2} \sin^2 \varphi = 1 - \left(1 - \frac{a^2}{b^2} \right) \sin^2 \varphi, \quad F = \frac{a^2}{c^2} \frac{1}{B} - 1$$

$$\int \sqrt{1+ax^2} dx = \frac{x}{2} \sqrt{1+ax^2} + \frac{1}{2\sqrt{a}} \sinh^{-1}(\sqrt{ax})$$

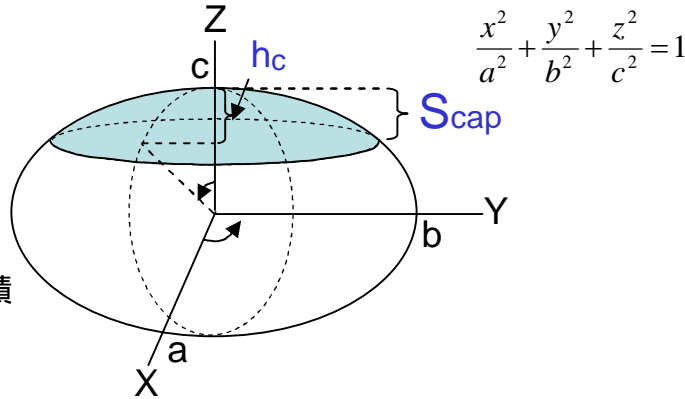
$$t = \sin \varphi, \quad d\varphi = \frac{dt}{\sqrt{1-t^2}}$$

$$B(t) = 1 - \left(1 - \frac{a^2}{b^2} \right) t^2, \quad F(t) = \frac{a^2}{c^2} \frac{1}{B(t)} - 1$$

$$f(t) = \sqrt{B(t)} \left(\sqrt{1+F(t)h^2} + \frac{\sinh^{-1}(h\sqrt{F(t)})}{h\sqrt{F(t)}} \right)$$

$$F(t) = 0 \Rightarrow f(t) = 2h \frac{a}{c}$$

楕円体の帽子部の表面積を数値積分で求める



帽子部の表面積

$$\begin{aligned}
 S_{cap} &= \iint dS \\
 &= \int_{\theta=0}^{\cos^{-1}h} \int_{\varphi=0}^{2\pi} \sqrt{b^2 c^2 \sin^2 \theta \cos^2 \varphi + c^2 a^2 \sin^2 \theta \sin^2 \varphi + a^2 b^2 \cos^2 \theta} \sin \theta d\theta d\varphi \\
 &= 4bc \int_{\varphi=0}^{\frac{\pi}{2}} \sqrt{B} \int_{u=h}^1 \sqrt{1+Fu^2} du d\varphi \\
 &= 2bc \int_0^{\frac{\pi}{2}} \sqrt{B} \left[u\sqrt{1+Fu^2} + \frac{1}{\sqrt{F}} \sinh^{-1}(\sqrt{Fu}) \right]_h^1 d\varphi \\
 &= 2bc \int_0^{\frac{\pi}{2}} \sqrt{B} \left\{ \sqrt{1+F} - h\sqrt{1+Fh^2} + \frac{1}{\sqrt{F}} \left(\sinh^{-1}(\sqrt{F}) - \sinh^{-1}(h\sqrt{F}) \right) \right\} d\varphi \\
 &= 2bc \int_0^{\frac{\pi}{2}} \sqrt{B} \left\{ \sqrt{1+F} - h\sqrt{1+Fh^2} + \frac{1}{\sqrt{F}} \sinh^{-1} \left(\sqrt{F} \left(\sqrt{1+Fh^2} - h\sqrt{1+F} \right) \right) \right\} d\varphi \\
 &= ab \int_{-1}^1 \left\{ 1 - hG(t) + \frac{c}{a} \sqrt{\frac{B}{F}} \sinh^{-1} \left(\frac{a}{c} \sqrt{\frac{F}{B}} (G(t) - h) \right) \right\} \frac{dt}{\sqrt{1-t^2}} \\
 &= ab \int_{-1}^1 \left\{ 1 - hG(t) + (G(t) - h) \frac{\sinh^{-1} \left(\frac{a}{c} \sqrt{\frac{F}{B}} (G(t) - h) \right)}{\frac{a}{c} \sqrt{\frac{F}{B}} (G(t) - h)} \right\} \frac{dt}{\sqrt{1-t^2}} \\
 &= ab \sum_{i=1}^n \frac{\pi}{n} f \left(\cos \left(\frac{2i-1}{2n} \pi \right) \right)
 \end{aligned}$$

$$x = a \sin \theta \cos \varphi$$

$$y = b \sin \theta \sin \varphi$$

$$z = c \cos \theta$$

$$\mathbf{r} = a \sin \theta \cos \varphi \mathbf{i} + b \sin \theta \sin \varphi \mathbf{j} + c \cos \theta \mathbf{k}$$

$$d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial \theta} d\theta \times \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \theta \cos \varphi & b \cos \theta \sin \varphi & -c \sin \theta \\ -a \sin \theta \sin \varphi & b \sin \theta \cos \varphi & 0 \end{vmatrix} d\theta d\varphi$$

$$= (bc \sin \theta \cos \varphi \mathbf{i} + ca \sin \theta \sin \varphi \mathbf{j} + ab \cos \theta \mathbf{k}) \sin \theta d\theta d\varphi$$

$$dS = |d\mathbf{S}| = \sqrt{b^2 c^2 \sin^2 \theta \cos^2 \varphi + c^2 a^2 \sin^2 \theta \sin^2 \varphi + a^2 b^2 \cos^2 \theta} \sin \theta d\theta d\varphi$$

$$u = \cos \theta, \quad du = -\sin \theta d\theta \quad h = 1 - \frac{h_c}{c}$$

$$B = \cos^2 \varphi + \frac{a^2}{b^2} \sin^2 \varphi = 1 - \left(1 - \frac{a^2}{b^2} \right) \sin^2 \varphi, \quad F = \frac{a^2}{c^2} \frac{1}{B} - 1$$

$$\int \sqrt{1+ax^2} dx = \frac{x}{2} \sqrt{1+ax^2} + \frac{1}{2\sqrt{a}} \sinh^{-1}(\sqrt{a}x)$$

$$\sinh^{-1} x - \sinh^{-1} y = \sinh^{-1} (x\sqrt{1+y^2} - y\sqrt{1+x^2})$$

$$t = \sin \varphi, \quad d\varphi = \frac{dt}{\sqrt{1-t^2}} \quad B(t) = 1 - \left(1 - \frac{a^2}{b^2} \right) t^2, \quad F(t) = \frac{a^2}{c^2} \frac{1}{B(t)} - 1$$

$$G(t) = \sqrt{h^2 + (1-h^2) \frac{c^2}{a^2} B(t)}$$

$$f(t) = 1 - hG(t) + \frac{c}{a} \sqrt{\frac{B}{F}} \sinh^{-1} \left(\frac{a}{c} \sqrt{\frac{F}{B}} (G(t) - h) \right)$$

$$F(t) = 0 \Rightarrow f(t) = 2(1-h)$$

Gauss-Chebyshev integration

$$S = ab \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = ab \sum_{i=1}^n w_i f(x_i) \quad w_i = \frac{\pi}{n}, \quad x_i = \cos \left(\frac{2i-1}{2n} \pi \right)$$